Electron Fluid in a Polywell - Take 2

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The first version of an electron fluid model started from the Vlasov and Maxwell equations and assumed a particle distribution function. In the process, I found that not including the magnetic field has a profound affect on the outcome of the math, and the physics is clearly wrong. The purpose of this version is to show that including the magnetic field is straight forward and quite interesting.

Let's start with the particle distribution function. As stated previously, it has a spatial component and an energy component. The energy component is fundamental - we expect fewer particles to be around higher energy and more particles to be around lower energy. Looking at the previous form we note that

$$f \propto e^{-\frac{E}{\mathrm{kT}}} \tag{1}$$

where E is the particle energy and kT is the average energy of the ensemble in terms of its temperature.

The Lagrange formulation of mechanics uses the difference of kinetic energy and potential energy. In equation 1) we have the sum of kinetic and potential. Using the Lagrangian form of electromagnetic fields and particles as given by J.D. Jackson "Classical Electrodynamics" we can write the total energy in terms of the particles velocity, mass, electric potential and magentic potential (page 574)

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + e\vec{v} \cdot \vec{A} - e\Phi \tag{2}$$

where $\vec{v}$ is the particle velocity, c is the speed of light, m is electron mass, e is the electron charge, Φ is electric potential, $\vec{A}$ is the magnetic vector potential and equation 2) is fully relativistic (and transformed into SI units).

Expanding the relativistic term for $v \ll c$ and ignoring the constant mc^2 we can write the total particle energy based on the Lagrangian as

$$E = \frac{m}{2}v^2 + e\vec{v} \cdot \vec{A} + e\Phi \tag{3}$$

If we compare this with equation (9) in the first incarnation of this writing, we notice that the magnetic field is now included as part of the particle's energy description. This will be very important in the gradiant with respect to velocity as we shall soon see.

The problem with potentials is that they must have a reference. With magnetic potential there is an arbitrary gauge transformation. For our purposes, with the assumption that the MaGrid does not have an oscillation (at least to begin with) both the Coulomb gauge and Lorentz gauge

$$\nabla \cdot \vec{A} + \varepsilon_0 \mu_0 \frac{\partial \Phi}{\partial t} = 0 \tag{4}$$

become equivelent since

$$\frac{\partial \Phi}{\partial t} = 0 \tag{5}$$

For the moment we also assume the MaGrid does not have a changing magnetic field so we can use the potentials to compute the electric and magnetic fields in the usual manner:

$$\vec{E} = -\nabla \cdot \Phi - \frac{\partial \vec{A}}{\partial t} \tag{6}$$

$$\vec{B} = \nabla \times \vec{A} \tag{7}$$

So for a static MaGrid we can isolate the electric and magnetic potentials from their respective fields. For the electron fluid, this is not so easy.

The magnetic potential can be computed from the definition (see "Principles of Electrodynamics", Schwartz, pg 143)

$$\vec{A}(\vec{r}, t) = \int d^3 r' \frac{\vec{J}(\vec{r}\,', t)}{|\vec{r} - \vec{r}\,'|} \tag{8}$$

where $\vec{r}$ is the position in space, $\vec{r}\,'$ is the position of current density $\vec{J}$ being integrated over and t is time. To compute this I'm going to do something a touch strange - I want to save the data in spherical coordinate space, but I want to keep track of the potential (and thus fields) in cartesian coordinates. The purpose is to be efficient with memory and to make calculations simple at the same time.

In version 1, equation (29) I show

$$\left|\vec{\mathfrak{r}} - \vec{\mathfrak{r}}\,'\right| = \sqrt{\mathfrak{r}^2 + \mathfrak{r}'^2 - \mathfrak{r}\mathfrak{r}'[\cos(\theta - \theta')(1 + \cos(\varphi - \varphi')) - \cos(\theta + \theta')(1 - \cos(\varphi - \varphi'))]} \tag{9}$$

where $\vec{\mathfrak{r}}$ is a unitless version of $\vec{r}$. For the MaGrid looking at the coil around the x axis it is clear that

$$\mathfrak{r}' = \sqrt{1 + R^2}$$

$$x = 1$$

$$y = R\cos\delta \tag{10}$$

$$z = R\sin\delta$$

and δ is an angle up from the (x, y) plane perpendicular to the x axis. The angles corresponding to the spherical coordinates of equation (9) are

$$\varphi' = \tan^{-1}\frac{y}{x} = \tan^{-1}(R\cos\delta) \tag{11}$$

$$\theta' = \sin^{-1}\frac{z}{\mathfrak{r}'} = \sin^{-1}\frac{R\sin\delta}{\sqrt{1 + R^2}} \tag{12}$$

Taking the current flow around the loop so that the resultant B field would be pointing into the Polywell we can write the cartesian form of the current density along the loop as

$$\vec{J} = R\sin\delta\,\hat{y} - R\cos\delta\,\hat{z} \tag{13}$$

Putting (9), (11), (12) and (13) into (8) we get (dividing out I_0 to make things dimensionless)

$$\vec{\alpha}_B(\vec{\mathfrak{r}})_{+x} = \int_0^{2\pi} R d\delta(\sin\,\delta\,\hat{y}\,-\,\cos\,\delta\,\hat{z})\Big\{\mathfrak{r}^2 + 1 + R^2 - \mathfrak{r}\sqrt{1+R^2}\Big[\cos(\theta - \sin^{-1}\frac{R\sin\delta}{\sqrt{1+R^2}})\big(1 + \cos(\varphi - \tan^{-1}(R\cos\delta))\big) + \cos(\theta + \sin^{-1}\frac{R\sin\delta}{\sqrt{1+R^2}})\big(-1 + \cos\big(\varphi - \tan^{-1}(R\cos\delta)\big)\big)\Big]\Big\}^{-1/2} \tag{14}$$

To compute the magnetic potential around the $-x$ axis we change $\varphi' \to \pi + \varphi'$ and flip the sign on the current in (13). To compute the potential around the y axis we change $\varphi' \to \frac{\pi}{2} + \varphi'$ and use

$$\vec{J} = R\sin\delta\,\hat{x} - R\cos\delta\,\hat{z} \tag{15}$$

and around the $-y$ axis we change sign of both terms in (15) and send $\varphi' \to \frac{3\pi}{2} + \varphi'$ with respect to equation (14).

Going around the z axis is much simpler, the angle θ' is now fixed and the variable of integration is φ' itself. The angle θ' is given by

$$\theta' = \tan^{-1}\frac{1}{R} \tag{16}$$

and the potential is given by

$$\vec{\alpha}_B(\vec{\mathfrak{r}})_{+z} = \int_0^{2\pi} R d\varphi'(\sin\,\varphi'\,\hat{x}\,-\,\cos\,\varphi'\hat{y})\Big\{\mathfrak{r}^2 + 1 + R^2 - \mathfrak{r}\sqrt{1+R^2}\Big[\cos\Big(\theta - \tan^{-1}\frac{1}{R}\Big)(1 + \cos(\varphi - \varphi')) + \cos\Big(\theta + \tan^{-1}\frac{1}{R}\Big)(-1 + \cos(\varphi - \varphi'))\Big]\Big\}^{-1/2} \tag{17}$$

which in principle can be done analytically. For the $-z$ axis we just flip sign since the current goes in the opposite direction.

In the first version I took

$$\bar{E} = geV = kT \tag{18}$$

and converted the energy to a dimensionless form relative to the MaGrid voltage. Since the magnetic potential is directly related to the magnetic field we have a similar transformation with

$$\vec{A}_B = \frac{\mu_0 I_0}{4\pi}\vec{\alpha}_B \tag{19}$$

and the subscript B refers to the MaGrid magnetic field so we can deal with the electron induced magnetic field later. The dimensionless form of equation (1) becomes

$$f(\vec{\mathfrak{r}}, \vec{\mathfrak{u}}, 0) \propto e^{-\frac{1}{g}\left(\mathfrak{u}^2 + \frac{\vec{\mathfrak{u}}\cdot\vec{\alpha}_B}{c_p} + \psi + \phi\right)} \tag{20}$$

where ψ is the MaGrid potential as described by equations (16) through (22) in version 1, $\mathcal{C}_p$ is given by

$$\mathcal{C}_p = \frac{2\pi}{\mu_0 I_0}\sqrt{\frac{2mV}{e}} \tag{21}$$

We can now add the a particle density distribution to equation (20) to give it spatial density as well as energy density and then proceed to plug this form of the particle distribution function into the velocity integrals of equations (1) and (2) in the "Electron Fluid" take one.

A numerical attempt at solving the electric potential generated by the particle distribution above failed spectacularly. To solve this problem, I need to introduce a constant related to the magnetic potential. If this constant is added to the Lagrangian it does not change the derivation of magnetic or electric fields because these real quantities are gradiants and curls of the potentials. Following the form of the particle distribution function described in "Electron Fluid in a Polywell", version 1, I write the full particle distribution as

$$f(\vec{\mathfrak{r}}, \vec{\mathfrak{u}}, 0) = \frac{R'(\alpha, \beta, s)}{e^{-\alpha\mathfrak{r}^2} + \beta\mathfrak{r}^2} e^{-\frac{1}{g}\left(\mathfrak{u}^2 + \frac{\vec{\mathfrak{u}}\cdot\vec{\alpha}_B}{\mathcal{C}_p} + \psi + \phi\right) - \frac{\alpha_{B\,\mathrm{Max}}^2 g}{4\mathcal{C}_p^2}} \tag{22}$$

Where $\alpha_{B\,\mathrm{Max}}$ is the maximum value found from computing the dimensionless magnetic potential in equations (8) through (17). The reason for this choice will be seen below.

The total charge density everywhere in the Polywell is given by the integral over velocity, and the potential everywhere is given by the integral over the charge density divided by the distance from each point in the fluid to the point in question. In math, the charge density is

$$\rho(\vec{r}, 0) = -\tilde{n}e\frac{R'(\alpha, \beta, s)}{e^{-\alpha\mathfrak{r}^2} + \beta\mathfrak{r}^2} e^{-\frac{\psi+\phi}{g} - \frac{\alpha_{B\,\mathrm{Max}}^2 g}{4\mathcal{C}_p^2}} \int d^3\mathfrak{u}' e^{-\left(\frac{\mathfrak{u}'^2}{g} + \frac{\vec{\mathfrak{u}}'\cdot\vec{\alpha}_B}{\mathcal{C}_p}\right)} \tag{23}$$

where

$$\tilde{n} = \frac{N_e}{\frac{4}{3}\pi s^3} \tag{24}$$

is the total number of electrons per Polywell volume and $-e$ by itself is the charge on an electron.

To do the integral over velocity we can create an arbitrary reference frame associated with the magnetic potential $\vec{\alpha}_B$ and take

$$\vec{\mathfrak{u}}' \cdot \vec{\alpha}_B(\vec{\mathfrak{r}}) = \mathfrak{u}'\,\alpha_B(\vec{\mathfrak{r}})\cos\nu \tag{25}$$

where ν is the angle between the velocity vector and the magnetic potential. The full integral over all velocity space is then given by

$$\rho(\vec{r}, 0) = -\tilde{n}e\frac{R'(\alpha, \beta, s)}{e^{-\alpha\mathfrak{r}^2} + \beta\mathfrak{r}^2} e^{-\frac{\psi+\phi}{g} - \frac{\alpha_{B\,\mathrm{Max}}^2 g}{4\mathcal{C}_p^2}}\, 2\pi \int_0^\infty \mathfrak{u}'^2 d\mathfrak{u}' e^{-\frac{\mathfrak{u}'^2}{g}} \int_0^\pi \sin\nu\, d\nu e^{-\frac{\mathfrak{u}'\alpha_B(\vec{\mathfrak{r}})\cos\nu}{\mathcal{C}_p}} \tag{26}$$

The factor of 2π comes from going around the magnetic potential line with the assumption that the velocity is not a function of this angle. The integral over ν is given in Gradshteyn & Ryzhik "Tables of

Integrals, Series and Products" form 3.915.1 and we get

$$\rho(\vec{r}\,,0) = -\,2\pi\tilde{n}e\frac{R'(\alpha,\beta,s)}{e^{-\alpha\mathfrak{r}^2}+\beta\mathfrak{r}^2}e^{-\frac{\psi+\phi}{g}-\frac{\alpha_{B\,\mathrm{Max}}^2 g}{4\mathcal{C}_p^2}}\int_0^\infty \mathrm{u}'d\mathrm{u}'e^{-\frac{\mathrm{u}'^2}{g}}\frac{2\mathcal{C}_p}{\alpha_B(\vec{\mathfrak{r}})}\sinh\frac{\mathrm{u}'\alpha_B(\vec{\mathfrak{r}})}{\mathcal{C}_p} \tag{27}$$

This integral is also possible to solve analytically from 3.562.3 and we find

$$\rho(\vec{r}\,,0) = -\,2\pi\tilde{n}e\frac{R'(\alpha,\beta,s)}{e^{-\alpha\mathfrak{r}^2}+\beta\mathfrak{r}^2}e^{-\frac{\psi+\phi}{g}-\frac{\alpha_{B\,\mathrm{Max}}^2 g}{4\mathcal{C}_p^2}}\frac{\alpha_B(\vec{\mathfrak{r}})\,g}{4\mathcal{C}_p}\sqrt{\pi g}\,e^{\frac{\alpha_B^2(\vec{\mathfrak{r}})}{4\mathcal{C}_p^2}g} \tag{28}$$

Let's recount where we are and how we got here. The assumptions so far are that we have a specific distribution of electrons in space along with an energy distribution based on the total potential plus kinetic energy derived from the Lagrangian which is kinetic minus potential energy. That this makes sense can be seen in Krall and Trivelpiece "Principles of Plasma Physics" where in chapter 7 (page 363) they give an example of constant magnetic field and show a constant of the motion depends on the magnetic potential.

Equation 27 makes clear why the choice of "arbitrary" constant is not so arbitrary in terms of actually performing a numerical calculation. The exponential term containing the magnetic potential is positive definite, and unless it is reduced to a reasonable level, is impossible to compute meaningfully with a computer.

The normalization constant $R'(\alpha,\ \beta,\ s)$ can be computed by integrating the charge density over the volume of the Polywell and setting the result equal to the total charge in the system. This can be written

$$Q = -\,\tilde{n}e\frac{(\pi g)^{\frac{3}{2}}}{2\mathcal{C}_p}\int_0^s \mathfrak{r}'^2 d\mathfrak{r}'\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta'\,d\theta'\int_0^{2\pi}d\phi'\frac{\alpha_B(\vec{\mathfrak{r}})R'(\alpha,\beta,s)}{e^{-\alpha\mathfrak{r}^2}+\beta\mathfrak{r}^2}e^{-\frac{\psi+\phi}{g}-\frac{g\left(\alpha_B^2(\vec{\mathfrak{r}})-\alpha_{B\,\mathrm{Max}}^2\right)}{4\mathcal{C}_p^2}} \tag{29}$$

Similar to equation (31) in version 1 of this paper, the dimensionless potential can be written

$$\phi_f(r,\theta,\varphi) = \frac{1}{4\pi\varepsilon_0 VL}\int\frac{d^3\mathfrak{r}'}{|\vec{\mathfrak{r}}-\vec{\mathfrak{r}}'|}\left\{-\,\tilde{n}e\frac{(\pi g)^{\frac{3}{2}}}{2\mathcal{C}_p}\frac{\alpha_B(\vec{\mathfrak{r}})R'(\alpha,\beta,s)}{e^{-\alpha\mathfrak{r}^2}+\beta\mathfrak{r}^2}e^{-\frac{\psi+\phi}{g}-\frac{g\left(\alpha_B^2(\vec{\mathfrak{r}})-\alpha_{B\,\mathrm{Max}}^2\right)}{4\mathcal{C}_p^2}}\right\} \tag{30}$$

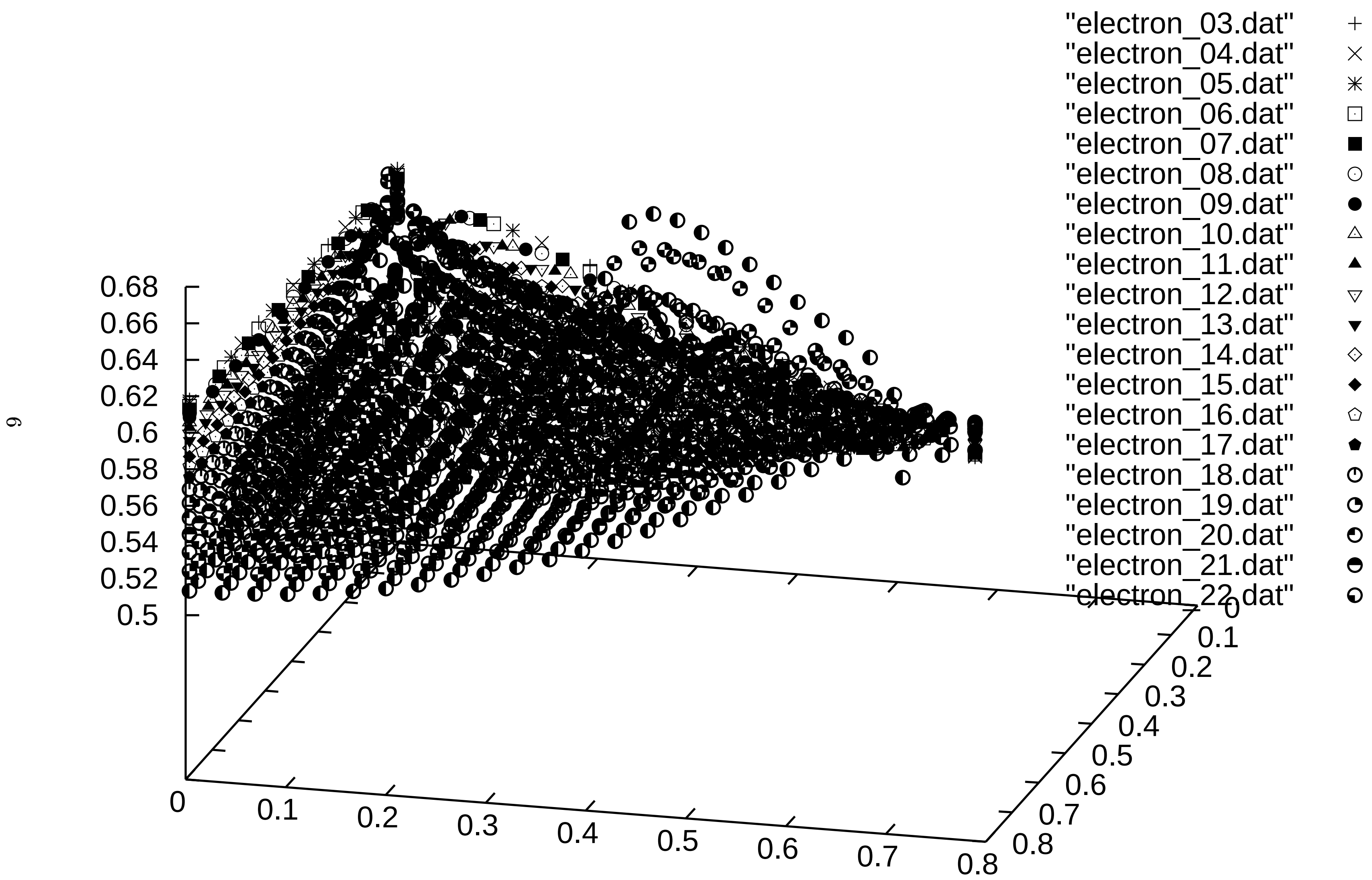
In order for this to be dimensionless, we can divide (30) by (29) to eliminate $R'(\alpha,\ \beta,\ s)$ and cancel out the charge by picking some arbitrary total Q. If we take the total charge on the MaGrid it is easy to show that

$$Q = 48\pi^2\varepsilon_0 VRL \tag{31}$$

The result is

$$\phi_f(r,\theta,\varphi) = 12\pi R\frac{\int\frac{d^3\mathfrak{r}'}{|\vec{\mathfrak{r}}-\vec{\mathfrak{r}}'|}\frac{\alpha_B(\vec{\mathfrak{r}})}{e^{-\alpha\mathfrak{r}^2}+\beta\mathfrak{r}^2}e^{-\frac{\psi+\phi}{g}-\frac{g\left(\alpha_B^2(\vec{\mathfrak{r}})-\alpha_{B\,\mathrm{Max}}^2\right)}{4\mathcal{C}_p^2}}}{\int d^3\mathfrak{r}'\frac{\alpha_B(\vec{\mathfrak{r}})}{e^{-\alpha\mathfrak{r}^2}+\beta\mathfrak{r}^2}e^{-\frac{\psi+\phi}{g}-\frac{g\left(\alpha_B^2(\vec{\mathfrak{r}})-\alpha_{B\,\mathrm{Max}}^2\right)}{4\mathcal{C}_p^2}}} \tag{32}$$

Thus the potential distribution must be self consistent and it depends on the voltage distribution in space from the MaGrid as well as the magnetic potential on the MaGrid. An "arbitrary" constant has been added to the magnetic potential to make the calculations possible. It remains to be seen if these calculations are feasible on a modern desktop computer.



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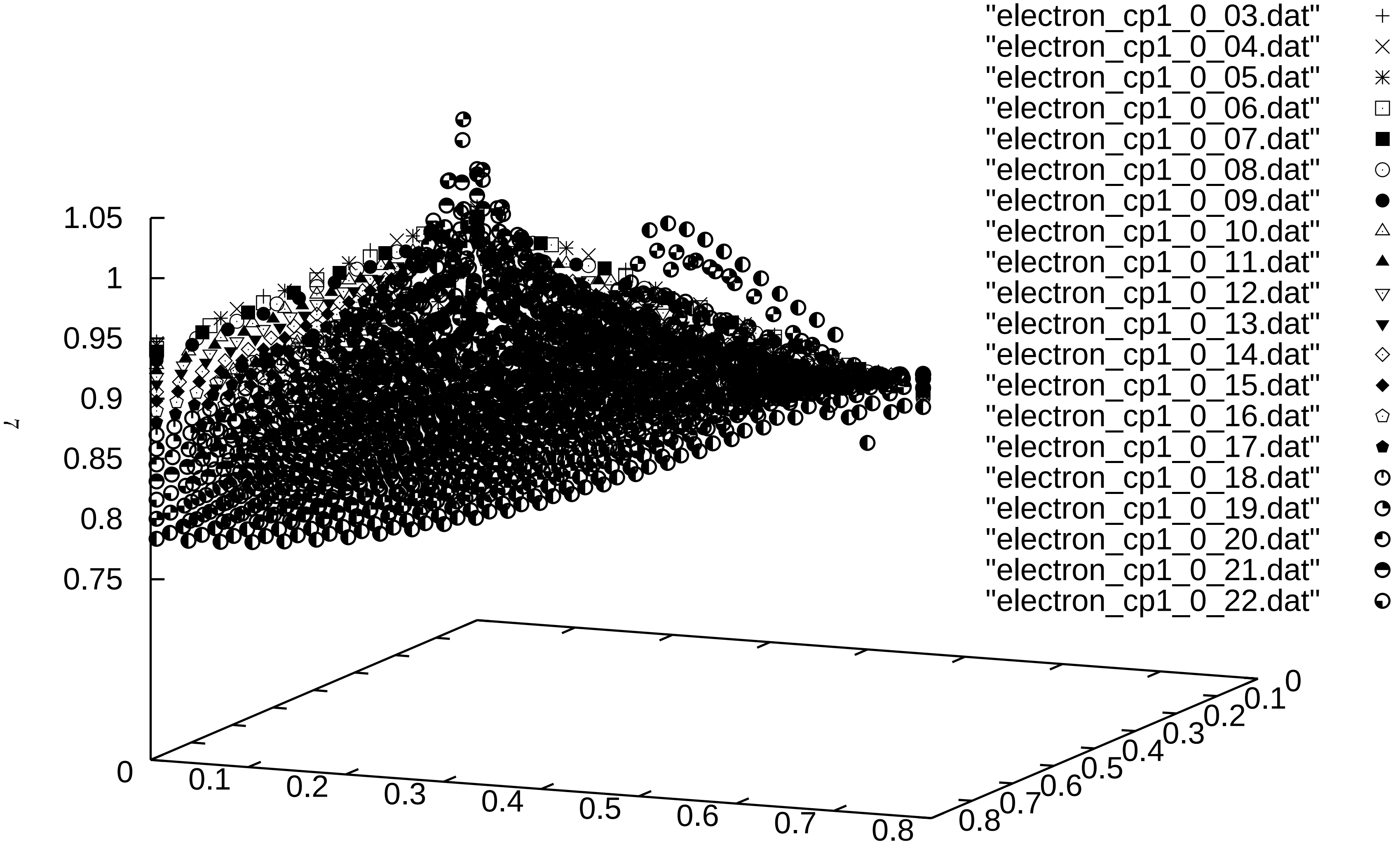

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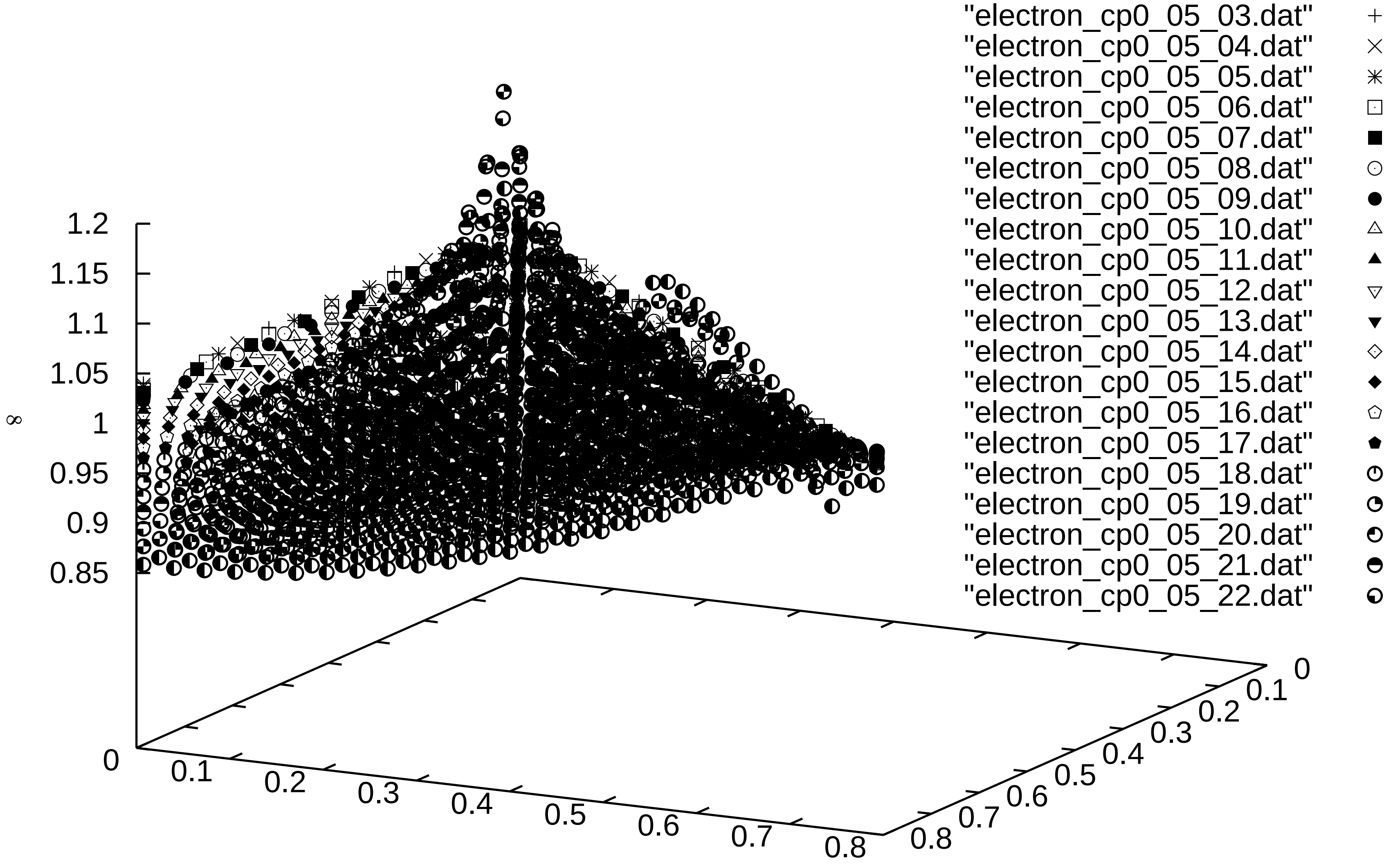

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There are a limited number of values for $\mathcal{C}_p$ for which the calculation of equation (32) can be carried out on a desktop computer. The first of three figures above has all references to α_B removed, it is purely an integral over the electrostatic and density portions (same as version 1 of this paper). The next two versions have α_B included. We note that the shape of the resulting potential is the same, the amplitude just shifts a little. The amplitude shift between no magnetic potential and including magnetic potential is a factor of two, but it still amounts to a potential shift. Since fields and forces depend on gradiants, this constant shift does not matter.

The middle figure above has $\mathcal{C}_p = 1.0$ and the last figure has $\mathcal{C}_p = 0.05$. The bottom integral of equation (32) varies from about 1.4 for the first case to about 10^{-27} for the second. Lower values of $\mathcal{C}_p$ are not computable with double precision floating point processors. Since realistic values of $\mathcal{C}_p$ are in the range of 10^{-5}, this formulation is useless for prediction.

Several assumptions went into the formulation of equation (32). One of them was that the local particle distribution was a Maxwellian function of velocity. Another is the form of particle distribution in equation (1). What the numbers say essentially is that the magnetic field does not matter for the electric potential. This is counter intuitive and probably wrong - particles flow along field lines and their presence determines the potential.