The Crossed Integers

(or, Fun with Arithmetic Modulo n)

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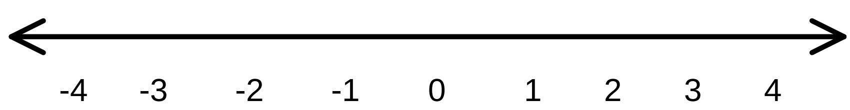
This note introduces a family of curious algebraic systems that I call *sets of crossed integers*. These systems generalize the system of integers modulo n (known as Z_n to students of abstract algebra). The crossed integers have algebraic properties that are strange compared to those of the integers modulo n.

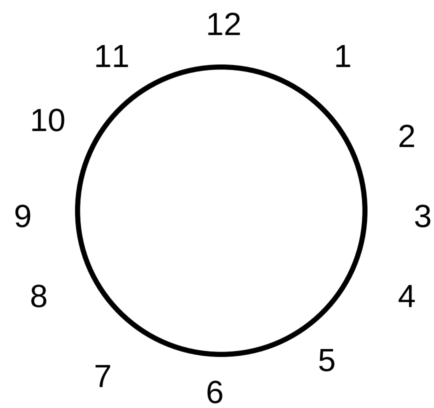
For the sake of readers who are new to abstract algebra, I'll begin by explaining arithmetic modulo n. Then I'll present the crossed integers and some of their antics.

First, Some Background: Clock Arithmetic

All of us are familiar with the set of integers: Z = {..., -2, -1, 0, 1, 2,... }. Beginning with the integers, it's possible to do other kinds of arithmetic besides the familiar arithmetic we learned in grade school. The most basic of these other kinds of arithmetic is sometimes called "clock arithmetic." The idea behind clock arithmetic is simple: instead of counting on a number line



we count on a clock face:



We do this by treating the number 12 as if it were the number 0. In clock arithmetic, the only numbers are 0 through 11, with the number 12 serving as 0. If you are counting, adding or subtracting, and you reach the number 12 or something greater than 12, then you subtract off 12 until you get an answer in the range 0 through 11. For example, in clock arithmetic:

5 + 5 = 10 (just as in ordinary arithmetic)
5 + 6 = 11 (ditto)
5 + 7 = 0 (because 12 is considered the same as 0)
5 + 8 = 1 (because 13 is one more than 12)
5 + 9 = 2 (because 14 is two more than 12)
etc.

Something similar happens with subtraction: for example,

3 - 1 = 2
3 - 2 = 1
3 - 3 = 0
3 - 4 = 11 (because 0 is considered the same as 12, so one less than 0 is 11)
3 - 5 = 10 (because two less than 0 is 10)
etc.

We can use any positive integer (not just 12) as a basis for a "clock" arithmetic. For example, we can treat 5, 3, or 22 as zero, instead of treating 12 as zero. This kind of generalized clock arithmetic is very well known to students of abstract algebra. It is called *arithmetic modulo n*, where n is the number that is treated as 0. The arithmetic of the clock face is arithmetic modulo 12.

Mathematicians have placed arithmetic modulo n on a firm foundation by regarding the "numbers" in that arithmetic as *residue classes* instead of as actual numbers. A residue class modulo n is a set of integers that differ from a fixed integer only by multiples of n. Thus, for example, the residue class of 1 modulo 12 is {..., -23, -11, 1, 13, 25, ...}. I won't go into the topic of residue classes here, because it would lead me into unnecessary technicalities. Readers interested in this topic can find it discussed in standard textbooks on abstract algebra.

The set of integers modulo n -- or, if one prefers, the set of residue classes modulo n -- is conventionally denoted by Z_n. Thus, the set of numbers of clock arithmetic (arithmetic modulo 12) is called Z_{12}.

From the standpoint of abstract algebra, the set Z_n, together with its operations of addition and multiplication, forms an algebraic structure known as a *ring*. The familiar set of integers Z, with its usual addition and multiplication, also is a ring. I won't discuss rings in detail here (consult your friendly neighborhood abstract algebra textbook for details).

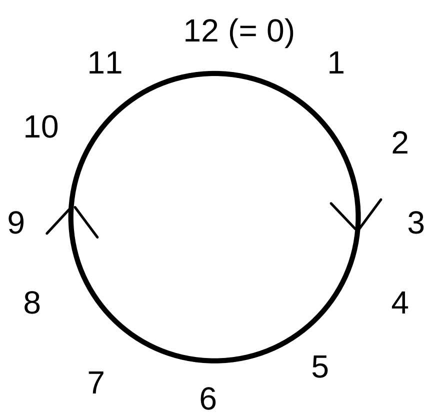
A Step Into Chaos: The Crossed Integers

When we think about the set of ordinary integers and the sets of integers modulo n, we tend to picture these sets as having certain *shapes*. The integers can be visualized as a number line, like the one I drew earlier in this note. The numbers in clock arithmetic (Z_{12}) are best visualized as forming a circle (the clock face). In general, the integers modulo n can be thought of as forming a circle. A circle is the natural way to draw a "number line" for Z_n. We can generate the appropriate shape for Z or for Z_n by drawing dots for the elements of the set and connecting each element x to its successor x + 1 with a line segment.

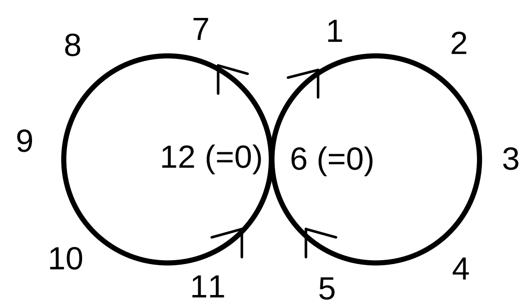
The ordinary integers can be pictured as a line, but by identifying some of the integers with each other and changing the rules of arithmetic accordingly, we get a circle instead of a line. While musing upon this fact, I had the following weird thought:

> **Is it possible to modify the integers even further, so that instead of forming a circle, they form a *figure eight*?**

Here is a pair of diagrams that show what I mean by this:



Integers modulo n
(specifically, n = 12)



"Crossed" integers (???)

The above diagram on the right represents a set of numbers in which 12 has been identified with 0, just as in Z_{12}. However, the number 6 *also* has been identified with 0. What is more, the numbers in the diagram have *not* been identified modulo 6; for example, 7 is not identical to 1. The identification of two different numbers with 0, independently of each other, is what makes the diagram a figure eight, instead of just a circle.

We could call these "numbers" the *crossed integers*.

There are many different ways to construct sets of crossed integers. The way shown in the diagram, in which 12 and 6 are separately identified with 0, is not the only way. We could have used other numbers instead of 12 and 6 (for example, 10 and 5). Thus, there are many systems of crossed integers, just as there are many systems Z_n of integers modulo n.

As is the case with Z_n, we can regard the crossed integers as sets of integers (generalized residue classes), instead of as single integers. I will sketch the way to do this in the Appendix to this note. (Treating the crossed integers as sets, analogous to residue classes, makes the "identification" of 12 and 6 with 0 less mysterious.)

To create Z_n from the ordinary integers, we had to identify n with 0. To twist the integers into a figure eight, we must do something very different. The above diagram leaves some questions unanswered. We identified both 12 and 6 with 0; we did not identify any other numbers with 0. If we start counting at 1 in the diagram, eventually we get to 6. After 6 (which is equal to 0), the next step forward is 1. Thus, the successor of 6 is 1. But in the diagram, there are two possible steps after 6: to 1, or to 7. To specify the arithmetic of the crossed integers shown in the diagram, we must answer the question: What is the relationship between 6 and 7, if it is not one of successorhood? In other words: why does 7 come after 6 in the diagram, when the successor of 6 is 1?

There are two "cheap" ways to answer this question:

(1) Let 6 have two successors. Then addition is a multiple-valued operation. If we take this path, then we abandon the assumption that addition is a binary operation. The idea of a binary operation lies at the heart of abstract algebra. We don't want to deviate this far from conventional algebra if we can help it.

(2) To ensure that each number has only one successor, identify each number's apparent successors in the diagram. Proceeding in this way, we identify 7 with 1, so that 6 has only one successor. But this only postpones the problem; the number 7, which now equals 1, now has two successors, 8 and 2. So to make successors unique, we must identify 8 = 2 also. But then 8 has successors 9 and 3.... It isn't hard to guess what will happen in the end: the two loops of the diagram will be identified with each other, and the structure we end up with will be the old familiar integers modulo 6 (also known as Z_6).

Besides these two relatively uninteresting answers, there is a more interesting way:

(3) Interpret the diagram so that one step in the direction of the arrows *doesn't necessarily indicate successorhood.* Don't assume that 7 is the successor of 6, even though 7 comes right after 6 in the diagram! Instead, assume that 1, and only 1, is the successor of 6. Then 7 isn't the successor of anything.

If we adopt this alternative, then a single step forward in the diagram doesn't always indicate the addition of 1. Usually it does -- but the step from 6 to 7 does *not* indicate the addition of 1. This choice allows us to answer the question about what is the successor of 6. The answer: 1 is the successor of 6, while 7 is not.

Choice (3) raises another question: Once this choice is made, and 7 is not a successor of 6, can we still visualize the system as a figure eight? It might seem that we've cut the bond between 6 and 7. We can't get to 7 by starting at 6 and adding 1. One can ask whether we can get to 7 at all by adding other numbers together. If we cannot do this, then the system is no longer picturable as a figure eight. But this is not a real threat. The fact that we can't get directly from 6 to 7 by adding 1 does *not* imply that we can't get from numbers *below* 6 to 7 by adding something. To preserve the figure-eight picture, all we need to do is insure that it's possible to get from the numbers before 6 (namely 1, 2, 3, 4, and 5), to the number 7, by adding something to the numbers before 6. For example, we might let $5 + 2 = 7$. Since 5 is one step before 6, this lets us visualize 7 as being one step after 6 in the diagram -- even though 7 is not equal to $6 + 1$. Thus, the figure eight structure is preserved -- even though its meaning has become somewhat less simple.

We have managed to convert the integers modulo 12 into a figure eight-shaped structure by identifying 6 with 0 and imposing the following constraints on addition:

$5 + 1 = 6$

$6 + 1 = 1$

BUT:

$5 + 2 \neq 1$ (instead, $5 + 2 = 7$)

Evidently, this system of numbers is going to have a very odd addition table! The first thing we notice is that the addition is not associative (at least if we assume that $1 + 1 = 2$):

$(5 + 1) + 1 = 6 + 1 = 1$

$5 + (1 + 1) = 5 + 2 = 7 \neq 1$

Thus, if $1 + 1 = 2$ in the crossed integers, then the crossed integers do not form a group, or even a semigroup, under addition.

Arithmetic in the Crossed Integers

What does the addition table of this structure look like? We already have decided that $0 + 1 = 1$, $5 + 1 = 0$, and $5 + 2 = 7$. We want to make addition in the crossed integers resemble the addition of integers modulo 12 as much as possible. Actually, we don't have to depart from the addition table of Z_{12} except for additions that involve 6 either as an addend or as a sum. For example, the addition facts for $1 + x$ could take the following form:

$1 + 1 = 2$

$1 + 2 = 3$

$1 + 3 = 4$

$1 + 4 = 5$

$1 + 5 = 0$ (different from Z_{12})

$1 + 0 = 1$

$1 + 7 = 8$

$1 + 8 = 9$

$1 + 9 = 10$

$1 + 10 = 11$

$1 + 11 = 0$

These choices are in accord with the spirit of our decision about $5 + 2 = 7$: if in Z_{12} we can add a number x before 6, plus some other number y, to get a number after 6, then the same addition performed in the crossed integers should give the same result.

We arrive at the following addition table:

+	0	1	2	3	4	5	7	8	9	10	11
0	0	1	2	3	4	5	7	8	9	10	11
1	1	2	3	4	5	0	8	9	10	11	0
2	2	3	4	5	0	7	9	10	11	0	1
3	3	4	5	0	7	8	10	11	0	1	2
4	4	5	0	7	8	9	11	0	1	2	3
5	5	0	7	8	9	10	0	1	2	3	4
7	7	8	9	10	11	0	2	3	4	5	0
8	8	9	10	11	0	1	3	4	5	0	7
9	9	10	11	0	1	2	4	5	0	7	8
10	10	11	0	1	2	3	5	0	7	8	9
11	11	0	1	2	3	4	0	7	8	9	10

Given this addition table, we can define multiplication as repeated addition in something like the usual manner. Let x and y be crossed integers. First define x&q, where q is a conventional integer in Z, inductively as follows:

x&1 = x

x&(q + 1) = (x&q) + x

Then define:

x*y = x&r

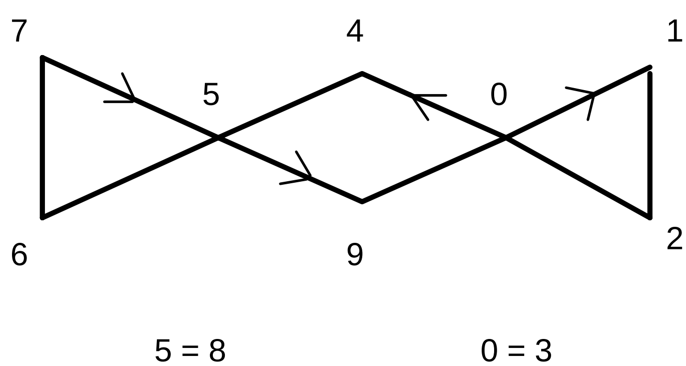
where r is the conventional integer corresponding to y. (That is, if y is the crossed integer 1, then r is the ordinary integer 1 -- and likewise with 1 replaced by 2, or 3, etc.)

I won't write out the multiplication table here.

General Crossed Integers

So far, we have studied only one system of crossed integers -- the one we obtained by taking the integers modulo 12 and identifying 6 with 0. More generally, we can take the integers modulo n, select some m < n, and identify m with 0. A possible notation for this system is $Z_{m\#n}$. In this notation, the system we have been studying is $Z_{6\#12}$. Our recipes for defining addition and multiplication can be generalized to $Z_{m\#n}$.

It may be possible to generalize this construction to get crossed integers with more than one crossover point. An example would be the structure that one gets from Z_{11} by identifying 5 = 8 and 0 = 3:



I will not attempt to perform this generalization here. This is a problem for future work.

Other questions for the future:

What happens when we try to generalize this construction from simple figure-eight shapes to arbitrary countable graphs?

More specifically: Given any countable connected graph, we can map 0 to a chosen node x, and then trace the graph forward and backward, mapping the rest of the integers (positive and negative) to the graph in such a way that each pair of consecutive integers map to neighboring nodes. (There may be a lot of doubling back and crossing over during this procedure, depending upon the details of the graph.) By identifying integers assigned to the same node under the mapping (i.e., by forming equivalence classes of integers under the equivalence relation of belonging to the same node), we can then create a set of crossed integers that reflect the shape of the graph -- just as our simple set of crossed integers reflected the shape of the figure-eight graph. What will the addition and multiplication tables of these systems be like?

APPENDIX. Representing crossed integers as generalized residue classes.

We can generalize the residue class construction of Z_n to encompass the crossed integers. I will do this with the crossed integers shown in the figure-eight diagram in this note.

To distinguish the crossed integers from the genuine integers (in Z), let us denote the crossed integer n by #n, and the genuine integer n (in Z) by n. Then we may equate the crossed integers #n to sets of genuine integers as follows. (Here the variables m, p and q range over Z.)

for n = 1, ..., 11,
#n = {q | for some m, q - n = 12m}

#0 = {q | for some m and p, q = 6m + 12p}
(the set of all integers that differ from 0 by an integer 6m + 12p,
where m and p are integers).

For n = 1, ..., 11, #n is just the residue class of n modulo 12 -- an object familiar from the standard textbook construction of Z_n. Further, #0 is just the residue class of 0 modulo 6, because 6 divides 12. (#0 would be more complicated if we had chosen numbers that are relatively prime instead of 6 and 12.)

In this way, we can regard crossed integers as sets, just as we can regard integers modulo n as sets. In identifying 12 and 6 with 0, we haven't really identified any distinct numbers at all; we've just formed some sets.