

The Folded Complex Numbers: A Mathematical Curiosity

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This note describes a way to construct a strange algebraic system that resembles, in some respects, the field \mathbb{C} of complex numbers. First I will describe this system intuitively. Then I will present a formal construction of the system. For reasons that will quickly become apparent, I call this system the set of *folded complex numbers*.

The Folded Complex Numbers: An Intuitive Picture

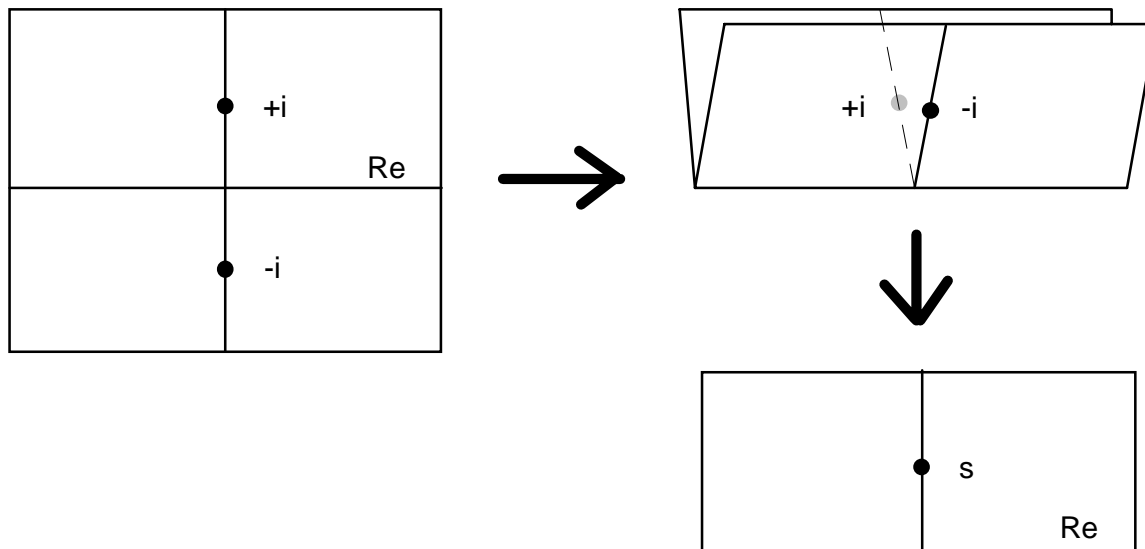
Intuitively, the folded complex numbers are what you get when you take the complex numbers and try to identify i with $-i$. In other words, the folded complex numbers are objects like the complex numbers, except that the "imaginary unit" i in the folded complex numbers obeys the following identity:

$$i' + i' = 0$$

Elements that obey this identity are nothing new to abstract algebra. For example, in the ring \mathbb{Z}_{2n} of integers modulo $2n$, we have $\underline{n} + \underline{n} = 0$, where \underline{n} is the residue class of n modulo $2n$.

The net result of this identification of i with $-i$ is a new system of generalized numbers in which each number $a + bi$ is identified with its complex conjugate $a - bi$. To avoid confusion, we will call the "imaginary unit" in the new system s instead of i . Thus, the new generalized numbers will be of the form $a + bs$, where a and b are real numbers, $s^2 = -1$ and $s + s = 0$.

Intuitively, the change from the complex numbers to this new system corresponds to folding the complex plane along the real axis:



Topologically speaking, the resulting folded structure is an orbifold -- though here we are more interested in this structure's algebra than in its topology. Later we will see that this folded structure is not necessarily the best way to visualize the new system. Nevertheless, this picture does illustrate what it means to identify i and $-i$.

The Folded Complex Numbers: A More Rigorous Approach

How do we construct, in a rigorous manner, this new system of generalized numbers? We will simply modify the well-known construction of the complex numbers, in which complex numbers are constructed as pairs of real numbers. Through a slight variation on this construction, we will obtain the folded complex numbers. In what follows, we adopt the standard notations \mathbb{Z} , \mathbb{R} and \mathbb{C} for the ring of integers and the fields of real and complex numbers, respectively.

To construct the new system, we follow one of the standard constructions of the complex numbers, with certain key changes. Here is a first try at such a construction (we'll amend it later). Begin with the set \mathbb{R}^2 of all ordered pairs (a, b) of real numbers. Let $s = (0, 1)$, and write the element (a, b) in the form $a + bs$. (Thus, $s = (0 + 1s)$.) Because s is its own negative, we must define addition of the pairs in such a way that $s + s = 0$, or in the new notation, $(0 + 1s) + (0 + 1s) = 0 + 0s$. We'd also like the addition of the pairs to be componentwise addition, just as in the complex numbers. That is, we'd like to have

$$(a + bs) + (c + ds) = (a + c) + (b + d)s \text{ for all pairs } (a, b), (c, d).$$

This rule for addition is copied from the arithmetic of the complex numbers. However, in the new system, we have to be very careful with this rule. Why? Because $s + s = 0$ implies that many different ordered pairs can represent the same generalized number! We have at once:

$$(0 + 2s) = (0 + 1s) + (0 + 1s) = s + s = 0 + 0s.$$

Therefore, for any pair $a + bs$,

$$a + (b + 2)s = (a + bs) + (0 + 2s) = (a + bs) + (0 + 0s) = (a + bs).$$

More generally, $(a + bs) = (a + (b + 2n)s)$ for any integer n .

One can get rid of this nonuniqueness by way of a quotient construction -- specifically, by taking the quotient $\mathbb{C}/2i\mathbb{Z}$ of the complex numbers by the additive normal subgroup $2i\mathbb{Z}$, and treating the resulting quotient group as the folded complex numbers. Then the folded complex numbers become cosets rather than pairs of reals. However, there is a simpler-looking, though ultimately equivalent, way to do this construction. Just redefine the folded complex numbers as the set $\mathbb{R} \times [0, 2)$ of pairs of reals with second component positive and less than 2, rather than as the set \mathbb{R}^2 of all pairs of reals. (Algebraically, this amounts to a choice of coset representatives.) Then

redefine addition of these generalized numbers so that the second component of the sum is taken modulo 2:

$$(a + bs) + (c + ds) = (a + b) + T(c + d)s$$

where the function T maps each real number x into the remainder of x modulo 2.

Thus, for example,

$$(1 + 1.6s) + (3 + 0.7s) = 4 + 0.3s \quad (\text{not } 4 + 2.3s)$$

Intuitively, this rule for addition amounts to the following: just add two generalized numbers as if the quantities were complex numbers, and if the "imaginary" component turns out to be outside the range $[0, 2)$, then replace it with its remainder modulo 2.

We can define multiplication analogously, by modifying the usual definition for the complex numbers:

$$(a + bs)(c + ds) = (ac - bd) + T(ad + bc)s$$

Note that this rule for multiplication does not follow from the quotient construction, which was a quotient of additive groups only.

Now we are in a position to define formally the folded complex numbers.

Definition. The *folded complex numbers* are elements of the set $FC = \mathbb{R} \times [0, 2)$, with addition and multiplication defined as follows:

$$(a, b) + (c, d) = ((a + c), T(b + d))$$

$$(a, b)(c, d) = ((ac - bd), T(ad + bc))$$

Using the notation $a + bs$ for (a, b) , the operations in FC are defined by:

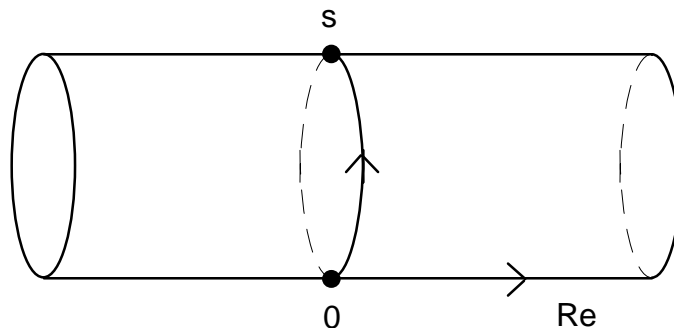
$$(a + bs) + (c + ds) = (a + c) + T(b + d)s$$

$$(a + bs)(c + ds) = (ac - bd) + T(ad + bc)s$$

As expected, we identify the real number a with $(a + 0s)$.

Folded Plane, or Cylinder?

Now let's go back to our earlier picture of the folding of the complex plane. Actually, that simple mental picture wasn't quite complete, because the construction doesn't lead to a half plane -- it leads to a cylinder. The real axis lies along the long direction of the cylinder; the circular direction corresponds to the interval $[0, 2)$ with its ends (0 and 2) identified.



This is rather surprising: we performed an operation on the complex plane that looked just like folding, but instead of a half-plane, we ended up with a cylinder!

Students of complex functions and geometry may find deeper meanings at this stage. Here I am

not concerned with these deeper meanings, but with this structure as an algebraic system, similar in some ways to the field of complex numbers. This structure has a known topology ($\mathbb{R}^1 \times S^1$), and resembles in some respects the direct product of the reals by the complex unit circle.

Topologically, this structure is an orbifold of a very simple sort. But my present concern is not topological. I want to look at certain *algebraic* properties of this structure.

Not a Ring... So What Is It?

If we look at addition in FC and ignore multiplication, then FC is an additive abelian group. This follows from the quotient construction mentioned above. The additive abelian group structure is nice, but once we start to look at multiplication, we find that many of the familiar rules of algebra are flouted. In particular, FC isn't a ring, because it doesn't obey the distributive laws. For example:

$$(1 + 1.95s) [(1 + 1.95s) + (1 + 1.95s)] = (1 + 1.95s)(2 + 1.9s) = -1.705 + 1.8s$$

$$[(1 + 1.95s) + (1 + 1.95s)] (1 + 1.95s) = (2 + 1.9s)(1 + 1.95s) = -1.705 + 1.8s$$

$$\begin{aligned} (1 + 1.95s)(1 + 1.95s) + (1 + 1.95s)(1 + 1.95s) &= (-2.8025 + 1.9s) + (-2.8025 + 1.9s) \\ &= -5.605 + 1.8s \end{aligned}$$

This structure isn't a field, or even a ring -- it's a mess!

However, it's interesting to note that the structure does behave like a ring (and, in fact, like the complex numbers) as long as one doesn't go too far from zero. This will be the case if one adds or multiplies elements with components small enough so that the remainder modulo 2 doesn't have to be taken during the calculation. That is, $a + bs$ and $c + ds$ will behave like complex numbers when added or multiplied together, provided that the following inequalities hold:

$$-2 < b + d < 2 \quad (\text{equivalent to } b + d < 2, \text{ since } b, d \text{ are positive})$$

$$-2 < ad + bc < 2$$

To graph these inequalities in the general case, we'd need 4 dimensions. However, if we fix a number $w = a + bs$, then we can use the (c, d) -plane to graph the numbers $c + ds$ that act like complex numbers when multiplied or added with w . We will call such numbers "complex-like" relative to w .

Consider the (c, d) -plane, with c as the horizontal (x) coordinate and d as the vertical (y) coordinate. In the following discussion it will be convenient to consider the entire (c, d) -plane, even though the actual numbers $c + ds$ all lie in the horizontal strip $0 \leq d < 2$. The inequality $b + d < 2$ tells us that the numbers that are complex-like relative to w all lie in the horizontal strip $0 \leq d < 2 - b$ (where $2 - b$ is positive because $b < 2$). Call this latter strip S . The complex-like numbers relative to w are precisely the numbers in S that obey the inequalities $-2 < ad + bc < 2$. We can distinguish four cases:

Case 1. $a \neq 0, b \neq 0$. Then the inequalities $-2 < ad + bc < 2$ reduce to:

$$-(b/a)c - 2/a < d < -(b/a)c + 2/a \text{ if } a > 0,$$

$$-(b/a)c + 2/a < d < -(b/a)c - 2/a \text{ if } a < 0.$$

which in turn condense into

$$-(b/a)c - 2/|a| < d < -(b/a)c + 2/|a|.$$

Thus, (c, d) must lie strictly between the lower line $d = -(b/a)c - 2/|a|$, and the upper line $d = -(b/a)c + 2/|a|$. Since both of these lines have the same nonzero slope, they define an infinite sloping strip, which must intersect the strip S somewhere. Thus, there will be points in S that satisfy both inequalities.

Case 2. $a \neq 0, b = 0$. The inequalities $-(b/a)c - 2/|a| < d < -(b/a)c + 2/|a|$ hold as in case 1, and reduce to $-2/|a| < d < 2/|a|$, which in S amounts to $d < 2/|a|$. Thus, some points (those with $d < \min(2 - b, 2/|a|)$) satisfy the inequalities.

Case 3. $a = 0, b \neq 0$. The inequalities $-2 < ad + bc < 2$ reduce to $-2 < bc < 2$, or to $-2/|b| < c < 2/|b|$. This defines a vertical strip which crosses S . Thus, some points in S satisfy the inequalities.

Case 4. $a = b = 0$. This is the trivial case of $w = 0$; the inequality is satisfied by all points in S .

A Concluding Question: Why Bother?

What is the point of all this? For now, the set of folded complex numbers is just a hunk of recreational mathematics. Those of us who are fascinated by the complex numbers and their generalizations (quaternions, Cayley numbers, etc.) should be happy to know that there is another way to generalize the complex numbers. Those of us who find the diversity of algebraic structures interesting should appreciate a structure that acts like a well-known ring in some spots and like a mess in others. Whether the folded complex numbers have any other significance is the biggest unanswered question about them!